

Non-minimal $\ln(R)F^2$ Couplings of Electromagnetic Fields to Gravity: Static, Spherically Symmetric Solutions

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(Dated: January 12, 2013)

Abstract

We investigate the non-minimal couplings between the electromagnetic fields and gravity through the natural logarithm of the curvature scalar. After we give the Lagrangian formulation of the non-minimally coupled theory, we derive field equations by a first order variational principle using the method of Lagrange multipliers. We look at static, spherically symmetric solutions that are asymptotically flat. We discuss the nature of horizons for some candidate black hole solutions according to various values of the parameters R_0 and a_1 .

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I. INTRODUCTION

We consider the non-minimal couplings of gravitational and electromagnetic fields described by a Lagrangian density. Such couplings may occur near compact astrophysical objects which has high mass density such as the neutron stars or black holes. The non-minimally coupled electromagnetic fields to gravity in RF^2 form were extended and classified [1, 2] to gain more insight into the relationship between space-time curvature and charge conservation. They were also obtained from a calculation in QED of the photon effective action from 1-loop vacuum polarization on a curved background [3] and from Kaluza-Klein reduction of a five-dimensional R^2 -Lagrangian [4, 5]. A non-minimally coupled Einstein-Maxwell Lagrangian in general may involve field equations of order higher than two. However, the type of non-minimal couplings we consider are obtained by the reduction of the Euler-Poincaré Lagrangian in five dimensions to four dimensions and involve at most second order field equations [6]. A three parameter family of non-minimally coupled Einstein-Maxwell field equations was studied in [7].

Recently, in the context of primordial magnetic fields present during the reheating epoch of the universe, the $R^m F^2$ -type couplings were discussed in [8]. The modified gravity with $\ln R$ terms [9] has also got much attention, since it could explain the observed acceleration of the universe and the dark energy concept without using any exotic fields. The behavior of the rotational velocities of test particles gravitating around galaxies is investigated in modified gravity in the context of non-minimal matter couplings [10, 11]. In this context, the modified $f(R)$ -Maxwell gravity with $I(R)F^2$ coupling terms [12, 13] and $f(G)$ -Maxwell gravity in which non-minimal coupling between electromagnetic fields and a function of Gauss-Bonnet invariant [14] were proposed to explain late-time cosmic acceleration. There are many other theories of gravity with non-minimal couplings [15], [16]. However, the non-minimal couplings with electromagnetic fields are not investigated in sufficient detail. Especially, finding spherically symmetric solutions is not an easy task for such theories [17],[18]. Furthermore, any arbitrary non-minimal coupling may not give rise to solutions satisfying physical asymptotic conditions and observations in solar and cosmological scales. Therefore, here we propose a non-minimal theory with a special $\ln R$ coupling term.

We first discuss a non-minimal $Y(R)F^2$ coupled Einstein-Maxwell theory using the algebra of exterior differential forms. We derive the field equations by a first order varia-

tional principle using the method of Lagrange multipliers. We choose in particular $Y(R) = \frac{1}{1-a\ln R/R_0}$ and look for static, spherically symmetric solutions that are asymptotically flat. In our model, the $Y(R)F^2$ coupled term in the Lagrangian leads to modifications both in the Maxwell and Einstein field equations. The modifications in the Maxwell equations can be related with the polarization and the magnetization in a specific medium. The non-minimal couplings also give important modifications to the structure of a charged black hole. These may shed light on some important problems of gravity such as dark matter and dark energy without introducing a cosmological constant or any other type of scalar fields.

II. FIELD EQUATIONS OF THE NON-MINIMALLY COUPLED THEORY

We will derive our field equations by a variational principle from an action

$$I[e^a, \omega^a_b, F] = \int_M L = \int_M \mathcal{L}^* 1, \quad (1)$$

where $\{e^a\}$ and $\{\omega^a_b\}$ are the fundamental gravitational field variables and F is the electromagnetic field 2-form. The space-time metric $g = \eta_{ab}e^a \otimes e^b$ with signature $(-+++)$ and we fix the orientation by setting $*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$. Torsion 2-forms T^a and curvature 2-forms R^a_b of spacetime are found from the Cartan-Maurer structure equations

$$T^a = de^a + \omega^a_b \wedge e^b, \quad (2)$$

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \quad (3)$$

We consider the following Lagrangian density 4-form;

$$L = \frac{1}{2\kappa^2} R * 1 - \frac{1}{2} Y(R) F \wedge *F, \quad (4)$$

where $\kappa^2 = 8\pi G$ is Newton's universal gravitational constant ($c = 1$) and R is the curvature scalar which can be found by applying interior product ι_a twice to the curvature tensor R_{ab} 2-form. We use the shorthand notation $e^a \wedge e^b \wedge \dots = e^{ab\dots}$, and $\iota_a F = F_a$, $\iota_{ba} F = F_{ab}$, $\iota_a R^a_b = R_b$, $\iota_{ba} R^{ab} = R$. The field equations are obtained by considering the independent variations of the action with respect to $\{e^a\}$, $\{\omega^a_b\}$ and $\{F\}$. The electromagnetic field components are read from the expansion $F = \frac{1}{2} F_{ab} e^a \wedge e^b$. We will confine ourselves to the unique metric-compatible, torsion-free Levi-Civita connection. We impose this choice

of connection through the constrained variations of the action by the method of Lagrange multipliers. That is, we add to the above Lagrangian density the following constraint terms:

$$L_C = (de^a + \omega^a_b \wedge e^b) \wedge \lambda_a + dF \wedge \mu, \quad (5)$$

where λ_a 's are Lagrange multiplier 2-forms whose variation imposes the zero-torsion constraint $T^a = 0$. We also use a first order variational principle for the electromagnetic field 2-form F for which the homogeneous field equation $dF = 0$ is imposed by the variation of the Lagrange multiplier 2-form μ .

The infinitesimal variations of the total Lagrangian density $L + L_C$ (modulo a closed form) are given by

$$\begin{aligned} \dot{L} + \dot{L}_C = & \frac{1}{2\kappa^2} \dot{e}^a \wedge R^{bc} \wedge *e_{abc} + \dot{e}^a \wedge \frac{1}{2} Y(R) (\iota_a F \wedge *F - F \wedge \iota_a *F) + \dot{e}^a \wedge D\lambda_a \\ & + \dot{e}^a \wedge Y_R (\iota_a R^b) (\iota_b F \wedge *F + F \wedge \iota_b *F) + \frac{1}{2} \dot{\omega}_{ab} \wedge (e^b \wedge \lambda^a - e^a \wedge \lambda^b) \\ & \dot{\omega}_{ab} \wedge \Sigma^{ab} - \dot{F} \wedge Y(R) *F + \dot{\lambda}_a \wedge T^a - \dot{F} \wedge d\mu. \end{aligned} \quad (6)$$

where $Y_R = \frac{dY}{dR}$, and the angular momentum tensor

$$\Sigma^{ab} = -\frac{1}{2} D[Y_R (F^{ab} *F + F^b \wedge \iota^a *F - F^a \wedge \iota^b *F - F \wedge \iota^{ab} *F)]. \quad (7)$$

The Lagrange multiplier 2-forms λ_a are solved uniquely from the connection variation equations

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = 2\Sigma_{ab}, \quad (8)$$

by applying the interior product operator twice as

$$\lambda^a = 2\iota_b \Sigma^{ba} + \frac{1}{2} \iota_{bc} \Sigma^{cb} \wedge e^a. \quad (9)$$

We substitute the λ_a 's into the e^a equations and after some simplifications we find the Einstein field equations for the extended theory as

$$\begin{aligned} \frac{1}{2\kappa^2} R^{bc} \wedge *e_{abc} + \frac{1}{2} Y (\iota_a F \wedge *F - F \wedge \iota_a *F) + Y_R (\iota_a R^b) \iota_b (F \wedge *F) \\ + \frac{1}{2} D[\iota^b D(Y_R F_{mn} F^{mn})] \wedge *e_{ab} = 0, \end{aligned} \quad (10)$$

while the Maxwell's equations read

$$dF = 0 \quad , \quad d*(YF) = 0. \quad (11)$$

In terms of a local inertial coordinate system (x^μ), the master equations (10) and (11) turn out to be equivalent to the following equations that were obtained by Bamba and Odintsov [13] by a variation procedure with respect to the metric:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}, \quad (12)$$

where the electromagnetic energy-momentum tensor components

$$T_{\mu\nu} = Y(g^{\alpha\beta}F_{\mu\beta}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) + \frac{1}{2}\{Y_R F_{\alpha\beta}F^{\alpha\beta}R_{\mu\nu} + g_{\mu\nu}g^{\gamma\sigma}\nabla_\gamma\nabla_\sigma[Y_R F_{\alpha\beta}F^{\alpha\beta}] - \nabla_\mu\nabla_\nu[Y_R F_{\alpha\beta}F^{\alpha\beta}]\}, \quad (13)$$

and the Maxwell equations

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}\tilde{F}^{\mu\nu}] = 0 \quad , \quad \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}YF^{\mu\nu}] = 0. \quad (14)$$

The effects of non-minimal couplings of the electromagnetic fields to gravity can be expressed by a constitutive tensor. For this purpose, Maxwell's equations satisfied by electromagnetic field F in an arbitrary medium can be written as

$$dF = 0 \quad , \quad *d*G = 0 \quad (15)$$

where G is called the excitation 2-form. We have taken the source-free Maxwell equations. The effects of gravitation and electromagnetism on matter are described by G . We can complete this system writing the following linear constitutive relation

$$G = \mathcal{Z}(F) \quad (16)$$

where \mathcal{Z} is a type-(2,2)-constitutive tensor. For the above theory, we have

$$G = Y(R)F. \quad (17)$$

We can further introduce the polarization 1-form $p \equiv d - e = (Y - 1)\iota_U F$ and magnetization 1-form $m \equiv b - h = (1 - Y)\iota_U * F$ relative to a time-like unit velocity vector field U of an inertial observer (For more details see [19–21]).

Now we consider the following function $Y(R)$ as a special case:

$$Y(R) = \frac{1}{1 - a_1 \ln \frac{R}{R_0}} \quad (18)$$

where a_1 is a dimensionless coupling constant and R_0 is a constant with the same dimension as that of R . We take $a_1 = 0$ to get back to the minimal Einstein-Maxwell theory. Furthermore, if we look at the limit $R \rightarrow R_0$ we again obtain the minimal Einstein-Maxwell case. It is interesting to note that in the limits as R goes either to zero or to infinity, the function $Y(R) \rightarrow 0$. That is, the effects of gravity become important and the electromagnetic effects can be neglected in these limits. On the other hand, as R approaches $R_0 e^{1/a_1}$, $Y(R)$ becomes very large so that the electromagnetic effects dominate. Finally, the function $Y(R)$ can be expanded as a power series in $\ln R$ for $a_1 \neq 0$ and $|a_1 \ln \frac{R}{R_0}| < 1$ as

$$\frac{1}{1 - a_1 \ln \frac{R}{R_0}} = \sum_{n=0}^{\infty} (a_1 \ln \frac{R}{R_0})^n. \quad (19)$$

In this form the model resembles the RG improved theory proposed by Bamba and Odintsov in Ref.[13]. Following this article, we try to relate the asymptotic freedom in a non-Abelian SU(2) gauge theory with a non-minimal Maxwell-modified gravity by setting

$$\frac{11\tilde{g}^2}{12\pi^2}\tilde{t} = \sum_{n=1}^{\infty} (a_1 \ln \frac{R}{R_0})^n \quad (20)$$

where \tilde{t} is a renormalization-group parameter, $\tilde{g}(\tilde{t})$ is the running SU(2) gauge coupling constant and $\tilde{g} = \tilde{g}(0)$. Therefore, if $|a_1 \ln \frac{R}{R_0}| \ll 1$, we obtain a similar kind of the RG parameter which has been proposed in [22].

III. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

We seek static, spherically symmetric solutions to the field equations which are given by the metric

$$g = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \quad (21)$$

and a static electric potential 1-form $A = V(r)dt$. Then

$$F = dA = V' dr \wedge dt = E dr \wedge dt. \quad (22)$$

A. Reduced Field Equations

After a lengthy calculation we reduce the non-minimally coupled Einstein-Maxwell equations (10) and (11) for the metric (21) and the electromagnetic 2-form (22) to the following

system of equations:

$$\begin{aligned}
\frac{1}{\kappa^2} \left(\frac{f^{2'}}{r} + \frac{f^2 - 1}{r^2} \right) - Y_R E^2 \left(\frac{f^{2''}}{2} + \frac{f^{2'}}{r} \right) + \frac{1}{2} Y E^2 - [(E^2 Y_R)' f] f - \frac{2}{r} f^2 (E^2 Y_R)' &= 0, \\
\frac{1}{\kappa^2} \left(\frac{f^{2'}}{r} + \frac{f^2 - 1}{r^2} \right) - Y_R E^2 \left(\frac{f^{2''}}{2} + \frac{f^{2'}}{r} \right) + \frac{1}{2} Y E^2 - (E^2 Y_R)' \left(\frac{f^{2'}}{2} + \frac{2f^2}{r} \right) &= 0, \\
\frac{1}{\kappa^2} \left(\frac{f^{2''}}{2} + \frac{f^{2'}}{r} \right) - Y_R E^2 \left(\frac{f^{2'}}{r} + \frac{f^2 - 1}{r^2} \right) - \frac{1}{2} Y E^2 - [(E^2 Y_R)' f] f - (E^2 Y_R)' \left(\frac{f^{2'}}{2} + \frac{f^2}{r} \right) &= 0,
\end{aligned} \tag{23}$$

$$Y E = \frac{q}{r^2}. \tag{24}$$

Here the curvature scalar

$$R = -f^{2''} - \frac{4}{r} f^{2'} - \frac{2}{r^2} (f^2 - 1). \tag{25}$$

q is the electric charge determined by the Gauss integral

$$\frac{1}{4\pi} \int_{S^2} *G = \frac{1}{4\pi} \int_{S^2} Y(R) E(r) r^2 \sin \theta d\theta \wedge d\phi = q. \tag{26}$$

B. Exact Solution

At this point, we will make a simplifying assumption and consider solutions that satisfy

$$R = \frac{C}{r^4} \tag{27}$$

where C is a constant to be fixed. Then, the field equations (23) and (24) simplify to

$$\frac{1}{2} \left(f^{2''} - \frac{2}{r^2} (f^2 - 1) \right) \left(\frac{1}{\kappa^2} + E^2 \frac{a_1}{R(1 - a_1 \ln \frac{R}{R_0})^2} \right) - \frac{qE}{r^2} = 0, \tag{28}$$

$$\frac{R}{2} \left(1 - \frac{a_1 \kappa^2 E^2}{R(1 - a_1 \ln \frac{R}{R_0})^2} \right) = 0, \tag{29}$$

$$\frac{1}{1 - a_1 \ln \frac{R}{R_0}} E r^2 = q. \tag{30}$$

Thus, for $a_1 \neq 0$, we can solve the unknown functions for the non-minimally extended Einstein-Maxwell theory with $C = a_1 \kappa^2 q^2$ and find

$$f^2(r) = 1 - \frac{2M}{r} + \frac{a_1 \kappa^2 q^2}{r^2} \ln \frac{r}{r_0} + \frac{\kappa^2 q^2 (1 + 5a_1)}{4r^2}, \tag{31}$$

$$E(r) = \frac{q}{r^2} + \frac{4qa_1 \ln \frac{r}{r_0}}{r^2}. \tag{32}$$

r_0 is an integration constant satisfying the relation $r_0^4 = \frac{a_1 \kappa^2 q^2}{R_0}$. The above solution is asymptotically flat. It is interesting to note that the electric field changes sign (due to

polarization effects) at $r = r_0 e^{-\frac{1}{4a_1}}$ that corresponds to the value $R = R_0 e^{1/a_1}$. On the other hand, the displacement vector field is an inverse square force field and hence does not change sign. We observe that both the curvature scalar and the quadratic curvature invariant $*(R_{ab} \wedge *R^{ab})$ for our solution are singular at the origin $r = 0$.

C. Horizons and Asymptotic Behavior

The roots of the metric function $f^2(r)$ are determined by the intersection points of a logarithmic curve and a parabola. In order to find these points we look at the numerator of the metric function:

$$n(r) = r^2 - 2Mr + \frac{\kappa^2 q^2}{4}(1 + 5a_1) + a_1 \kappa^2 q^2 \ln \frac{r}{r_0} = r^2 f^2(r). \quad (33)$$

The derivative of the function or $n'(r)$ has two local extrema at

$$r_1 = \frac{M}{2} + \frac{1}{2}\sqrt{M^2 - 2a_1 \kappa^2 q^2} \quad , \quad r_2 = \frac{M}{2} - \frac{1}{2}\sqrt{M^2 - 2a_1 \kappa^2 q^2}. \quad (34)$$

We find the number of event horizons depending on the critical values of the parameters a_1 and r_0 by looking at the behavior of the derivative of the above expression (33). Thus, $n'(r)$ has one root (r_1) for $a_1 < 0$, two roots for $0 < a_1 < \frac{M^2}{2\kappa^2 q^2}$, one root for $a_1 = \frac{M^2}{2\kappa^2 q^2}$, ($r_1 = r_2$) and no roots for $a_1 > \frac{M^2}{2\kappa^2 q^2}$. The number of the horizons will differ according to the sign of the function $n(r)$ at these critical points and the values of a_1 and r_0 . We will determine these intervals and plot the graphs of the function $n(r)$ in the interval for certain values of the parameters as follows.

Case A: For $a_1 < 0$, we can determine the interval of r_0 related with the numbers of horizons from the following inequalities (for a specific case see FIG.1);

1. If $n(r_1) < 0$, there are two horizons.
2. If $n(r_1) = 0$, there is one horizon.
3. If $n(r_1) > 0$, there is no horizon.

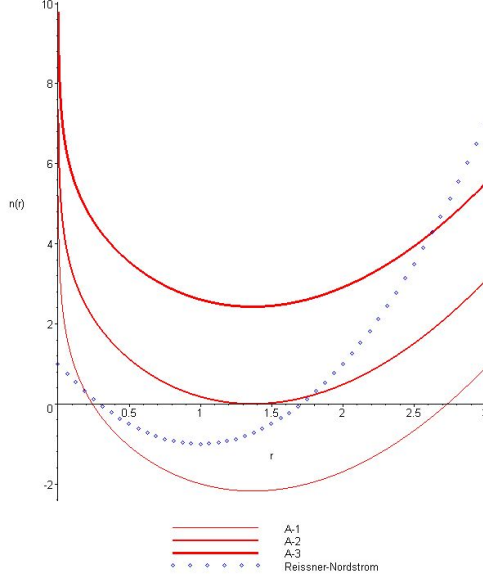


FIG. 1: The graph of the function $n(r)$ for the Case:A with $a_1 = -1$, A-1) $r_0 = 1$, $n(r_1) \simeq -2.17$, A-2) $r_0 \simeq 8.83$, $n(r_1) = 0$, A-3) $r_0 = 100$, $n(r_1) \simeq 2.43$ and Reissner-Nordström case is plotted by the dotted curve ($\kappa = M = q = 1$).

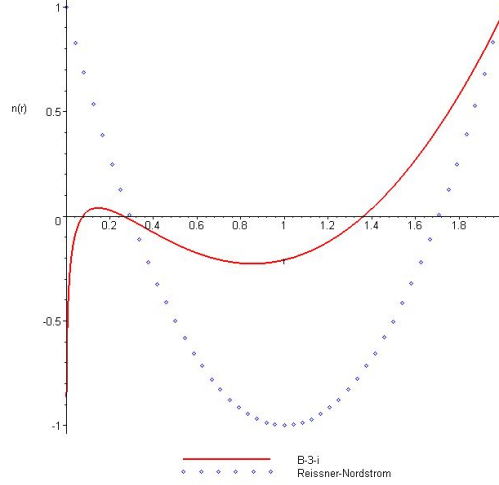
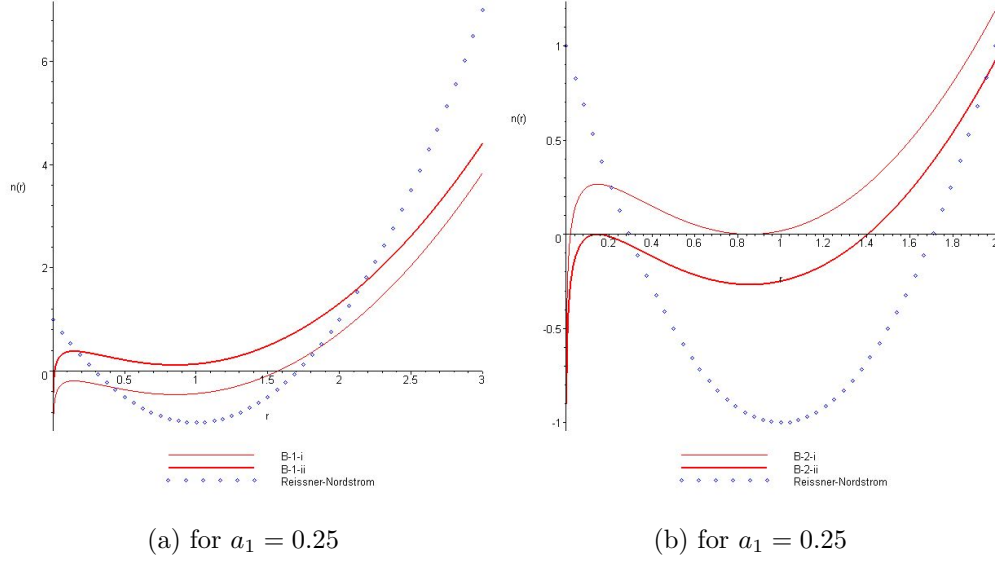
Case B: For $0 < a_1 < \frac{M^2}{2\kappa^2 q^2}$, (see FIG.2);

1. If i) $n(r_1), n(r_2) < 0$ or ii) $n(r_1), n(r_2) > 0$ there is one horizon.
2. If i) $n(r_1) = 0$ and $n(r_2) > 0$ or ii) $n(r_1) < 0$ and $n(r_2) = 0$ there are two horizons.
3. If i) $n(r_1) < 0$ and $n(r_2) > 0$ or ii) $n(r_1) > 0$ and $n(r_2) < 0$ there are three horizons. But it is not possible to find an r_0 in the second interval $n(r_1) > 0$ and $n(r_2) < 0$.

Case C: For 1) $a_1 = \frac{M^2}{2\kappa^2 q^2}$ and 2) $a_1 > \frac{M^2}{2\kappa^2 q^2}$ (see FIG.3); $n(r)$ increases monotonically from $-\infty$ to ∞ and there is one horizon.

The case with no horizon exhibits a naked singularity. The cases with a single horizon resemble the extreme Reissner-Nordström solution, while the cases with two horizons correspond to the the generic Reissner-Nordström geometry. The case with three horizons seems to be new.

Further investigation of the black hole properties can be done numerically for each of the cases above. We plan to deal with this problem in a future work.



(c) for $a_1 = 0.25, r_0 = 0.4$

FIG. 2: The graphics of $n(r)$ for the Case-B, B-1-i) $r_0 = 1$, $n(r_1) \simeq -0.45$, $n(r_2) \simeq -0.19$, B-1-ii) $r_0 = 0.1$, $n(r_1) \simeq 0.12$, $n(r_2) \simeq 0.38$, B-2-i) $r_0 \simeq 0.16$, $n(r_1) = 0$, $n(r_2) \simeq 0.26$, B-2-ii) $r_0 \simeq 0.47$, $n(r_1) \simeq -0.26$, $n(r_2) = 0$, B-3-i) $n(r_1) \simeq -0.22$ and $n(r_2) = 0.04$ and Reissner-Nordström case is plotted by the dotted curve ($\kappa = M = q = 1$).

IV. CONCLUSION

We considered a non-minimally $Y(R)F^2$ -coupled Einstein-Maxwell theory and looked for static, spherically symmetric solutions for a specific function $Y(R) = \frac{1}{1 - a_1 \ln \frac{R}{R_0}}$ that shows charge screening effects. We obtained a class of asymptotically flat solutions that include new

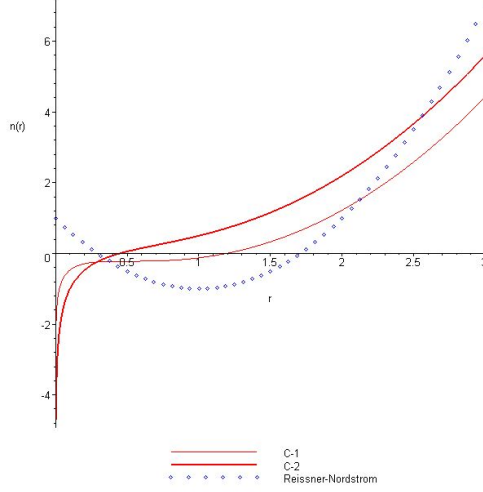


FIG. 3: The graph of $n(r)$ C-1) $a_1 = 0.5, r_0 = 1, n(r_1) = n(r_2) \simeq -0.22$, C-2) $a_1 = 1, r_0 = 1$ and Reissner-Nordström case is plotted by the dotted curve ($\kappa = M = q = 1$).

black hole candidate configurations, except for the parameter values $a_1 < 0$ and $n(r_1) > 0$ when there is a naked essential singularity at the origin. These particular solutions may shed light on some problems of gravity such as dark matter and dark energy without introducing a cosmological constant or any other exotic fields. This means that, if dark matter is not some strange matter, but, for instance the non-minimal couplings produce such effects [23]; then the electromagnetic potentials get modified at large (astrophysical) scales and thus contribute to the conventional electromagnetic energy density which may then be interpreted as the effects of dark matter. We also note that the electric charge q need not have a large value to have observable effects. If q is small, a_1 may be large so that the product $a_1 q^2$ becomes important at large (astrophysical) scales. This can explain the rotation curves of galaxies for certain parameter values [23].

V. ACKNOWLEDGEMENT

T.D. gratefully acknowledges partial support from The Turkish Academy of Sciences (TUBA). The research of Ö.S. is supported in part by a grant from The Scientific and Technological Research Council of Turkey (TUBITAK).

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